Understanding The Basic Thermal Properties Of SMT Devices

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Introduction

Ensuring optimal RF performance (that is, mechanical, thermal and electrical performance) of a surface-mount technology (SMT) ceramic device requires the careful procurement and use of consistently well-behaved materials, the use of proper and rigorous design principles, and proper attention paid to the appropriate parameters (on the part of the manufacturer in its design, and on the part of the user in its operation).

In this way, the reflection, dissipation, and transmission signatures of the device in question remain consistently and predictably well behaved. Some instances of devices that are designed to maximize dissipative loss are terminations, loads, absorptive filters and dividers, etc. Typically, these devices must minimize reflective loss (regardless of the nature of the signals at their inputs), which, when combined with proper dissipative design, results in an optimal transmission signature as well.

SMT devices that are non-dissipative by design are hybrid couplers, power dividers, filters, etc. Filters, of course, are designed such that some signals are to experience a minimum of reflection and dissipation while maximizing transmission, whereas other signals are to experience a maximum of reflection while minimizing transmission.

Other dissipative devices are attenuators, voltage dropping resistors, resistive splitters, etc. Attenuators must be designed such that reflective power is minimized and the remaining dissipative and transmitted power levels are ratioed accordingly to what is desired.

Again, the degree to which attention is paid to device RF performance parameters such as:

A. How well the thermal coefficients of expansion (TCEs) between device and board are matched,
B. How consistent the materials are in the construction of the device,
C. What design principles are used in the device’s construction,

D. What thermal management techniques are used to appropriately direct the device's dissipated energy (once placed), and
E. What circuit level methods are used to minimize reflective losses,

among others, all critically affect the RF performance of the SMT device. Optimal performance occurs when these devices are each properly matched (mechanically, electrically, and thermally) to the system in which they are placed.

A complete analysis of the RF performance parameters of a typical SMT device mounted in a general circuit board environment is beyond the scope of this paper, which will focus primarily on thermal considerations.

In addition, this paper will address three different situations in which power capacity for an SMT device can be optimized through the appropriate use of thermal conduction principles:

1. Basic thermal management on the circuit board to which an alumina SMT device is mounted,
2. Implementation of improved processing methods on the part of the SMT device itself, which keeps the device cooler (when under power) than when standard methods are applied, and
3. The use of aluminum nitride, rather than alumina, as the substrate for the devices, thus improving heat conduction significantly.

Basic thermal principles

Ceramic SMT devices (like all others) make use of the following thermal transport mechanisms when under power:

- Radiation,
- Convection (if not in an hermetically sealed environment or in outer space), and
- Conduction

It can be shown using basic physics and engineering mathematics that thermal conduction is the primary mode of heat transport for SMT ceramic devices, when conduction is properly accounted for in the device's operation. This is partly due to the fact that some ceramics (such as alumina, aluminum nitride, beryllium oxide, etc) possess thermal conductivities that approach the same order of magnitude as those of some metals. In addition, the mechanical strengths and TCEs of these same ceramics are close in magnitude to those of the commonly used circuit boards to which they will be attached. Other parameters such as density, dielectric constant, and dielectric loss are also appropriately desirable.

Because conduction is the dominant transport mechanism, it becomes critical to take advantage of this by accounting for it when designing and using most ceramic SMT

devices. Making good use of conductive design principles and applying them to the device's operation in the circuit will then optimize its overall RF performance, whether it is a device whose dissipative loss is designed to be small (like a "lossless" filter or divider) or large (like a termination) or otherwise an ordered fraction of the expected transmitted power (as in an attenuator).

Refer to *Figure 1*. Typical SMT devices are, by their nature, characterized by having a thickness that is much smaller than either of their other dimensions. This is fortunate for it allows for Fourier's heat law to be effectively applied, which serves as a good approximation for the device's conductive cooling capacity and thus the device's power capacity.

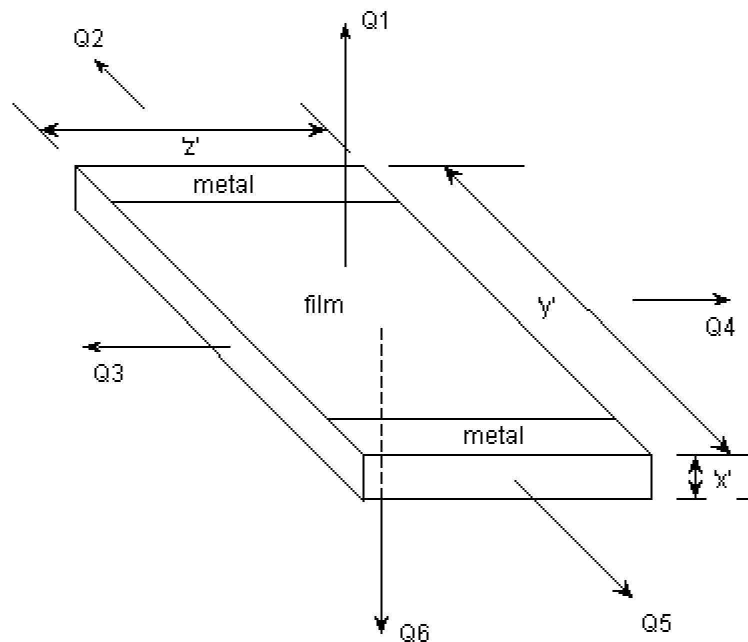


Figure 1: *Typical form for an SMT device. Q1 to Q5 represent potential heat flow directions when the transport mode is either radiative or conductive. Conductive heat transport occurs in the direction of Q6, when the circuit is appropriately designed. Typically, the device thickness (x) is much smaller than the other two dimensions (y or z).*

When the assumption is made that steady-state conductive thermal transport conditions are appropriate for a given performance analysis, the ease with which power capacity for an SMT device can be defined and calculated also becomes apparent.

Before looking at specific examples of a typical SMT device's thermal properties, a review of the basic principles of thermal physics (i.e.; radiation, convection, and conduction) is in order.

Radiation

Refer to *Figure 2*. The Stefan Boltzmann Law of Radiation states that the spectral radiance of a thermally emitting "black body" depends primarily on its temperature. When the spectral radiance (R) is integrated over all wavelengths (λ), the result takes the form:

$$\int R(\lambda, T)\, d\lambda \equiv \pi \int 2\, h\, c^2/\lambda^5\, (e^{hc/\lambda kT} - 1)^{-1}\, d\lambda$$

where λ is integrated from zero to infinity. The left-hand side becomes the radiant flux (Φ, measured in watts/meter^2), and the right side computes to σT^4.

So:

$$\Phi = \sigma T^4$$

For non-black bodies, this equation becomes:

$$Q_r/A = \varepsilon\, \sigma\, T^4$$

where Q_r is in watts; ε is a material-dependent constant that represents the degree to which the body approaches thermally emitting perfection (a black-body radiator has an emissivity of 1); σ is the Stefan-Boltzmann constant, which itself depends on the Boltzmann constant (k_B), Planck's constant, and the speed of light in a vacuum; T is the temperature of the radiating surface (in °K); and A is the area of the emitting surface (in m^2).

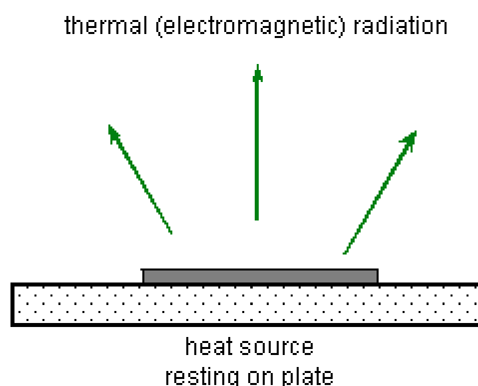


Figure 2

A quick example can be made with a slab of alumina at 150(C) and an area consistent with that of a 2512 resistor chip ($A \sim 20 E^{-6} m^2$):

ε = 0.68 - 0.88 (depending on wavelength)
$\sigma = 5.67 E^{-8} (W/m^2 K^4)$
T = 150(C) = 423 (K)

The result in watts is approximately:

Q_r = 32mW (radiated upward)

Alternatively, this computes to about 1600 W/m^2 = 150 W/ft^2 = about 1W per square inch. The sides of the slab add a small amount to the 32mW figure whose magnitude depends of course on the thickness of the slab. If we were also to include the bottom of the slab, which contributes another 32 mW, the top and bottom areas add up to 64 mW. This value consists of about 77% of the total radiated power when the slab thickness is 25 mil.

Note that this is a simple example that does not take into account any complexities like a resistive film, metal pads, glass overcoats, etc., which would be consistent with an SMT device. We are simply assuming that the entire radiating surface is alumina.

Convection

Convection is a complicated process, so we will only consider the very simplest of applications founded on basic principles.

Generally speaking, convection is a mode of thermal transport that involves the movement of molecules (unlike either radiation or conduction), where the molecules serve as agents of energy transfer. In most cases, air is the medium of transport, and convective heat transfer is only a fraction of the radiation heat transfer, the degree to which depends highly on the application. It is important to state that what we are speaking about is *natural* convection from a flat surface that is at constant temperature and is surrounded by air at atmospheric pressure. The term “natural” is used to distinguish from air that is moving by external means.

If we consider air at atmospheric pressure and temperature in a typical SMT application, it is reasonable to conclude that the area of the device being cooled must play the most significant role (as is the case with radiation).

The basic equation for heat flow where convection is the mode of cooling is given by:

$Q_h = h A (\Delta T)$

where it is assumed that the average heat flow through an area (A) is proportional to the temperature difference between the device and the air around it. The term (h) is a proportionality constant whose value depends itself on the temperature difference ΔT and the orientation that the device presents to its environment. For instance, given a horizontal surface and the above conditions, the primary numerical coefficient of h varies between about 0.4 and 0.6, depending on whether the surface is vertically oriented or horizontally oriented, respectively. We will use the larger value, since most SMT devices are designed to present a horizontal surface to the surrounding air.

Typically, the form of h is as follows:

$$\mathbf{h = b\ E^{-4}\ (\Delta T)^{1/4}}$$

where b is the primary numerical coefficient. In the above case, the units for h are cal/s cm^2 8K.

Values for h for a variety of applications are published as tables in some of the literature. In a planar application whose temperature delta is 150(C) to 70(C), we have a value for h as:

$$\mathbf{h = .6\ E^{-4}\ (4.184\ J/cal)\ (80)^{1/4}}$$ whose units are: W/cm^2 °K

Multiplying by 10^4 to get W/m^2 °K, we obtain:

$$\mathbf{h = 7.51\ W/m^2\ °K}$$

Then the expected value for heat dissipated for a given area is:

$$\mathbf{Q_h/A = h\ (\Delta T) = 7.51\ X\ 80 = 600\ W/m^2}$$

This converts to about 55W per square foot, or 388mW per square inch.

This assumes natural convection, as noted previously, in that the air is relatively still such that its motion is only inspired by the temperature difference, rather than the employment of a fan (forced convection). Were the application characterized by forced air convection, the velocity of the air, density of the air, and temperature of the air would each play a critical role in determining the Q/A result.

Conduction

Refer to *Figure 3*, where a block of material is shown subjected to a heat source on one end. Energy flows from the hotter temperature to the cooler temperature by means of molecular interaction (without the molecules themselves flowing). This is otherwise known as heat conduction. This condition can be a fairly good representative of the

environment presented to an SMT device under power while mounted to a circuit board, provided that the temperature of the baseplate is kept constant.

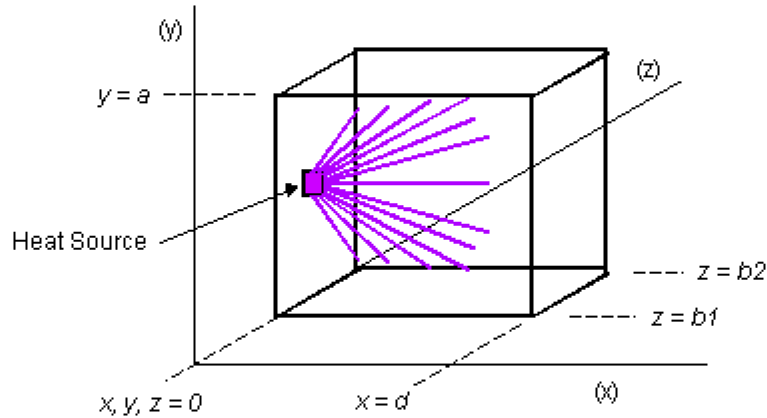


Figure 3: *The block conducts heat in every direction to its sides to a degree that depends on the divergence of the temperature gradient between the source and any given side.*

Fourier's Law of heat conduction states that the net heat flow through a region of area A (A = the cross-sectional area of the material equivalent to $\Delta y \Delta z$ = a (b2-b1) in *Figure 3*) is proportional to the temperature gradient between the endpoints:

$$Q/A = -k\ (\delta T_x/\delta x) \qquad (1)$$

or more succinctly

$$Q/A = -k \cdot \mathrm{grad}\ (T_x) \qquad (1a)$$

or just

$$Q/A = -\ k\ \nabla(T_x) \qquad (1b)$$

where k is a proportionality constant dependent on the material itself (ceramic material in our case). In fact, k is known as the thermal conductivity. In most cases, the thermal conductivity is not constant but is itself a function of temperature, and sometimes direction, for a given ceramic material.

Figure 4 shows the situation in one dimension. The heat flow is in the x direction, perpendicular to the area of the heat source. Note that similar equations to 1a or 1b can be formulated for the y and z dimensions. It is assumed for illustrative purposes that the material is thermally isotropic, resulting in a k that possesses the same value regardless of direction.

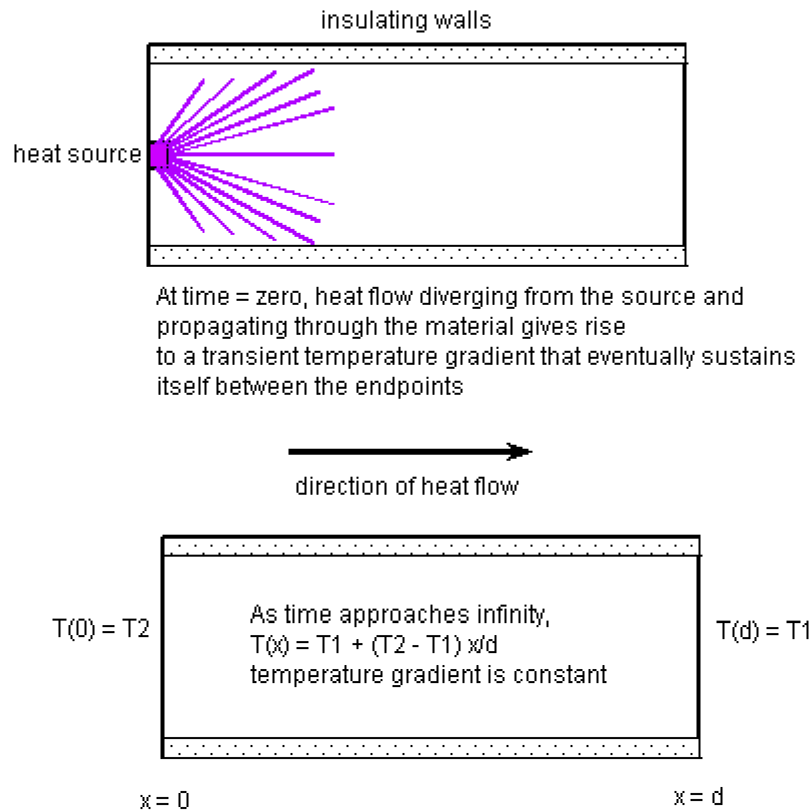


Figure 4

Once we have discovered a reliable form for the temperature function (T_x) and its derivative, we can solve this equation and determine how to apply it in a conductive SMT situation.

The boundary conditions for this problem shall be characterized by the following, to maintain simplicity:

a. There exists no heat exchange in the other two dimensions (y and z), which suggests that the other four faces are insulating, and
b. That the temperatures at the endpoints are kept constant. (at x = 0, the temperature is T1, while at x = d, the temperature is T2, for instance).

These boundary conditions will be critical in aiding our ability to find a form for T_x so that equation 1 can be effectively used.

The general differential equation for conductive energy transport is:

$$C_p (\delta T/\delta t) = - \text{div} (k \text{ grad } T)$$

or

$$C_p \, \delta T/\delta t = -\nabla \bullet (k \, \nabla T) \quad (2)$$

which states, in words, that the divergence of the temperature gradient within a region must be equivalent to the energy density that gives rise to the heat flow itself. The term on the right uses the scalar product (or dot product). This equation essentially follows from conservation of energy principles. (For a derivation of this, refer to the literature.) The proportionality constant k is defined as it was in equation 1.

After some mathematical manipulations, including a conversion to one-dimensional flow as indicated in *Figure 4*, a form for the temperature function (T) can be derived and the result is:

$$T_x = (T2 - T1)\, x/d + T1$$

from which its derivative (with respect to x) can be easily extracted:

$$\nabla T = \delta T/\delta x = (T2 - T1)/d$$

which can be substituted back into equation 1. The rest of the derivation is left to the Appendix.

Equation 1 then reads:

$$Q/A = -k \, (T2 - T1)/d \quad (2a)$$

so that for a block of alumina (k = 25 W/m °K) presenting an area (A) to the direction of perpendicular heat flow through a distance, d (= 1 inch = 25.4mm) and temperature difference of, say, 10(°K):

$$Q/A = 9{,}800 \text{ W/m}^2 = 900 \text{ W/ft}^2 = \text{about 6W per square inch}$$

The fact that the value of d affects the outcome to a large degree is one key to understanding why conduction plays a primary role in cooling, especially with typical SMT devices. The thickness of a particular heat-generating body, especially when that thickness is relatively small, only marginally helps a body cool through convection or radiation, whereas a small d dimension enhances conduction cooling.

In addition, a small d also makes the application of Fourier's Law (equations 1 and 2a) more accurate and appropriate.

To see why, refer to *Figures 3* and *4* again. The block conducts heat in every direction to its sides. However, if one face is kept at a constant temperature (lower than T2), we can consider the faces, back, top, and bottom to be insulating surfaces so that little heat flow occurs through them (i.e., in the y or z directions). Then a substantial temperature gradient occurs only from x = 0 to x = d over the area A = a [b2-b1].

Under these conditions, Fourier's Law becomes a relatively accurate approximation to a degree that depends on the relative magnitudes of d (thickness) versus the other two dimensions.

The approximation for SMT devices assumes a rectangular volume, such that one of the dimensions in the volume is very small compared with the other two. Under this 'pseudo-planar' condition, four of the surfaces are practically neglected, and most of the energy is assumed to be transferred from one (large area) side through the material to the other side. In the examples to follow, the results of this approximation will be confirmed.

Some examples

Figure 5 shows a typical application of an SMT device mounted to a circuit board. We will now investigate and compare examples of heat flow for a typical SMT ceramic substrate (of 25 mil thickness) using the three modes of cooling presented: conduction, radiation, and convection. We will use Fourier's Law for the conduction example:

$$Q_r = \varepsilon \sigma A (T2^4 - T1^4) \quad \text{(Radiation)}$$
$$Q_h = h A (\Delta T) \quad \text{(Convection)}$$
$$Q_c = -k A (\Delta T)/d \quad \text{(Conduction)}$$

where Q_c, Q_h, and Q_r are in watts, A is the area in each and is therefore equal for each example, and the material constants are as defined previously.

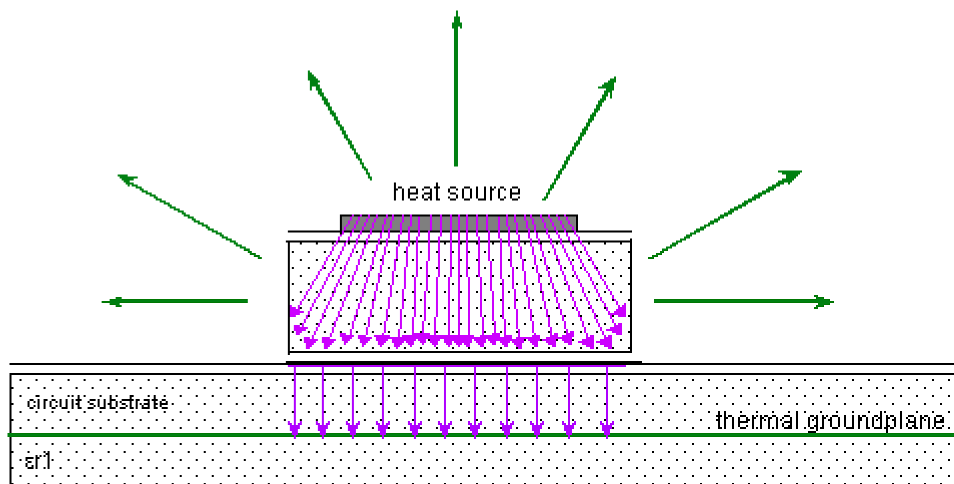


Figure 5: *Conduction occurs in the direction of the violet arrows to the baseplate of the chip, provided that the baseplate temperature is cooler than the heat source itself. In order for conduction to efficiently continue, the baseplate must be thermally connected to the thermal groundplane. For best results T_{ground} must be kept constant. Radiation and convection occur in the direction of the green arrows, provided that the air surrounding the chip is cooler than the heat source. SMD industry standards allow for full rated power when temperature of film does not exceed 150(C) and when the temperature of the surrounding air does not exceed 70(C).*

Starting with the first example as we concluded previously:

Q_r = 32mW

When the following are used:

ε = 0.88 for alumina
$\sigma = 5.67\ E^{-8}\ (W/m^2\ K^4)$
$A \sim 20\ E^{-6}\ m^2$,
T2 = 423 (K) and T1 = 0

This number is much less when the temperature of the surrounding air (T1) is taken into account (say at 70(C) = 343(K)). In that case, the 32mW is reduced to about 18mW.

As we also learned previously, the sides and bottom of the chip add to this, and the result carried forward is that the 36 mW consists of about 77% of the total potential radiated power, obtained by simply comparing relative areas.

In the convection case, we plug in the associated values of:

$h = 7.44\ J/°K{\cdot}m^2$

using the assumption that the surrounding (still) air is at a temperature of 343(K), the result in watts is approximately:

$Q_h = 7.44\ (20\ E^{-6})\ (423 - 343) = 12mW$

It is apparent that five sides can contribute to convection cooling, so that the total convective heat flow is closer to 15mW if the same assumptions are made about the sides of the chip as were made in the radiation case. Remember that for a vertical orientation (the four sides of the device), the value of h is less than that for the horizontal orientation. Moving air will drastically improve this number as well.

In any case, these calculations show that the total theoretical power transferred as a result of both radiation and convection for a 2512 slab of alumina (25 mil thick) at T = 150(C) in still air is at best about 60 mW.

When conduction is the mode of cooling, a baseplate temperature will be assumed for the value of T1, where the value of T2 is still 150(C) = 423(K).

If a 10-degree difference is used, the following result obtains for Q_c:

Q_c = about 8W

which is magnitudes greater than that for the other two cases. In fact, both radiation and convective cooling can still occur if the surrounding air is still at 343(K), so that the theoretical ratio of conductive to (radiative + convective) cooling for these examples is:

8.06/.06 ~ 134

This shows that conduction is a much more efficient cooling mechanism than either of the other two. Truly, no one should be surprised at this result, especially given the fact that most SMT ceramics have a relatively high thermal conductivity and that most surface-mount devices are quite thin, which make them much less likely to be able to take advantage of convection than a standard SIP or DIP, whose profiles are designed to present a large vertical area to passing air currents.

In addition, simply imagining the difference between holding one's hand a few inches over an active stove burner versus placing one's hand directly on the burner is intuitively instructive in itself.

Of course, it is not necessarily the case that a chip component, when mounted on top of a circuit board, exhibits absolutely *no* conductive cooling when appropriate thermal management techniques are not applied. Nevertheless, it is clear that making use of conductive principles when designing the circuit board for placement of a SMT device is to the designer's great benefit.

Actual power capacity of a chip termination of this size (2512) is drastically reduced from the 8W above due to a host of factors, one of which is the actual size of the heat generating area being much smaller than $20\ E^{-6}\ m^2$.

According to internal proprietary tests made by IMS on a group of chip terminations (whose baseplates were kept at constant temperatures for five distinct temperatures between 50(C) and 120(C)), the actual power capacity of a 2512 chip manufactured from 25 mil alumina (measured at currents of 1A and sometimes greater) is about 3W, when the temperature delta is 10 degrees.

Points to ponder

Refer to *Figure 6* below.

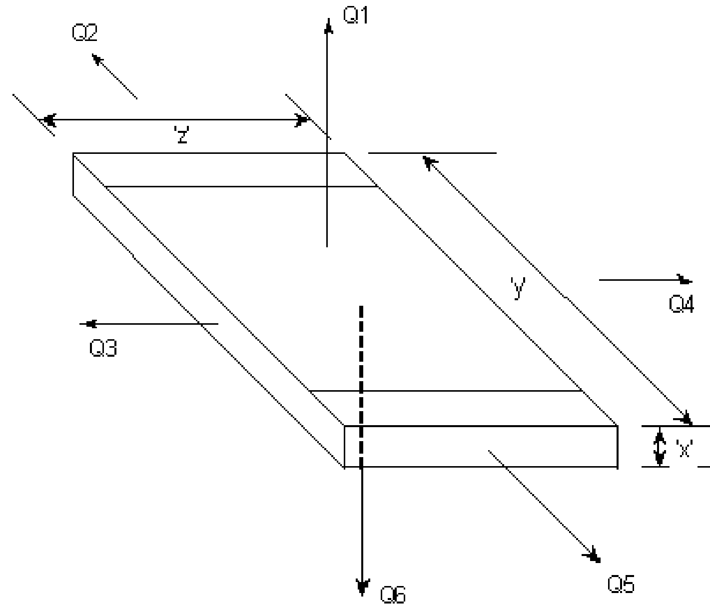


Figure 6: *In a non-conductive application, Q6 is assumed zero, and the total approximate thermal energy transferred away from the chip is*

Qn = Q1 + Q2 + Q3 + Q4 + Q5 *(radiative and convective)*

In a conductive application, the total thermal energy transferred away from the chip is

Qc = Q6 *(through conduction)* + Sum (Q1-Q5) *(radiative and convective)*

The smaller the thickness of the chip (x) compared with its area dimensions (z and y), the larger benefit conduction plays in thermal cooling, the more accurate is the application of Fourier's Law to the situation, and so the larger the ratio: Qc/Qn.

Note that, when a ceramic substrate's thickness is kept constant and the area of the top and bottom faces of the chip is allowed to decrease, the amount of conductive cooling that can take place is a smaller and smaller fraction of the whole. Put another way, a chip with a very small area and a certain thickness (d), will not be able to take advantage of conduction cooling to the same degree that a chip with a larger area (and same d) would. To put some numbers to this, take for instance a 0805 chip versus a 2512 chip, as before. The ratio of conductive cooling to (radiative + convective) was about 134 for the 2512.

When a 0805 chip is considered, its theoretical area is about 2.6 E^{-6} m^2 (versus 20 E^{-6} m^2 for the 2512 chip), but because its thickness is still 0.685 mm (25 mil), its total radiative capacity (taking into account five sides) is about 2.34mW / 28% = 8.4 mW. The convective power capacity is similarly 1.6mW / 38% = 4mW, making the total power dissipated under radiative and convective conditions to be about 12-13 mW.

By contrast, the conductive capacity using the same environmental factors as before computes to about 1W.

The conductive ratio for the 0805 chip is then 1/.013 = 77, significantly less than that for the 2512 chip, formulated above. Therefore, the benefit of conduction is expected to decrease for a given device thickness when the area of the chip in question decreases. This also should not be surprising, and this fact was borne out in the proprietary tests. This effect is shown to a small degree in the power derate curve depicted in *Figure 8*.

Methods of improving conduction for SMT devices

Decreasing thermal resistance

One consequence of Fourier's Law is the concept of thermal resistance. Clearly the thermal performance of a SMT device is partially governed by the size of and temperature level manifested by the film on its top side. Equation 2a can be written by rearranging terms:

$$\Delta T = Q\,(d/kA) = Q\,\Theta_r \qquad \text{(2b)}$$

The term: $\Theta_r \equiv d/kA$ is dependent on the material and is otherwise known as the thermal resistance of the element (of area A and thickness d). Note the electrical analogue:

$$\Delta V = I\,(\rho L/A) = I\,R$$

What follows from this is that for a given temperature difference (ΔT), the act of either reducing d or increasing A will result in a larger heat flow (Q), or simply a higher power capacity for the element in question.

At IMS, a proprietary method of processing SMT devices was implemented which led to a more even temperature distribution of the SMD film. The result was an effective increase in the film area (which generates heat when the device is under power). This led directly to a decrease in Θ_r, thus increasing the power capacity of the SMT device without increasing its actual size.

Another way of interpreting this is that the lower Θ_r of these devices leads to a smaller temperature rise (when compared with standard SMT devices of the same type) when the device is under power. These devices are members of the Super RCX family of chip resistors and terminations.

Aluminum nitride

A second consequence of Fourier's Law and equation 2b is that the power capacity of a device can also be increased by using a material whose thermal conductivity (k) is larger. This is the case with SMT devices manufactured from AlN when compared with those made from alumina (Al2O3).

Other materials are also used, one of which is beryllium oxide (BeO). This material is becoming less popular due to its potentially hazardous composition in the powder state.

Its use is also somewhat undesirable due to the fact that its dielectric constant is significantly different from that of Al_2O_3 (6.5 vs. 9.8, respectively). This makes replacing Al_2O_3 chips with BeO chips with the intent of increasing thermal conductivity a little complicated at and above RF frequencies, whereas replacing them with AlN chips results in a more tolerable dielectric constant adjustment of 9.8 to about 8.9

Provided that the processing materials used in the manufacture of the SMT device are compatible with ALN to the same or better degree that they are with alumina, a device may be fabricated whose power capacity is greatly enhanced over others simply by virtue of the fact that ALN has a higher k value than does Al_2O_3.

In fact, most ALN ceramics have a k value that ranges from 170 to 200 (W/m°K), which is 6 to 7 times that of the k value of alumina. Therefore, under identical conditions, the thermal resistance of a typical SMT device is reduced by a factor of 6 to 7 when ALN is the chosen substrate.

The other pertinent material coefficients of ALN are quite close to those of alumina as well, so that the overall RF performance of an ALN device is not sacrificed with its use as a good thermal conductor.

Typically, ALN ceramics have dielectric constants ranging between 8.5 and 9.5 depending on frequency, temperature, and the processing techniques employed to fabricate the ceramic. Another parameter is the loss tangent ($\tan\delta$) of ALN, which ranges between 0.004 and 0.008 (compare to Al_2O_3, whose $\tan\delta$ is about 0.0004). This helps matters when the device is intended to be resistive but likely makes ALN an undesirable ceramic to use (as opposed to Al_2O_3) when manufacturing lossless devices such as dividers and filters.

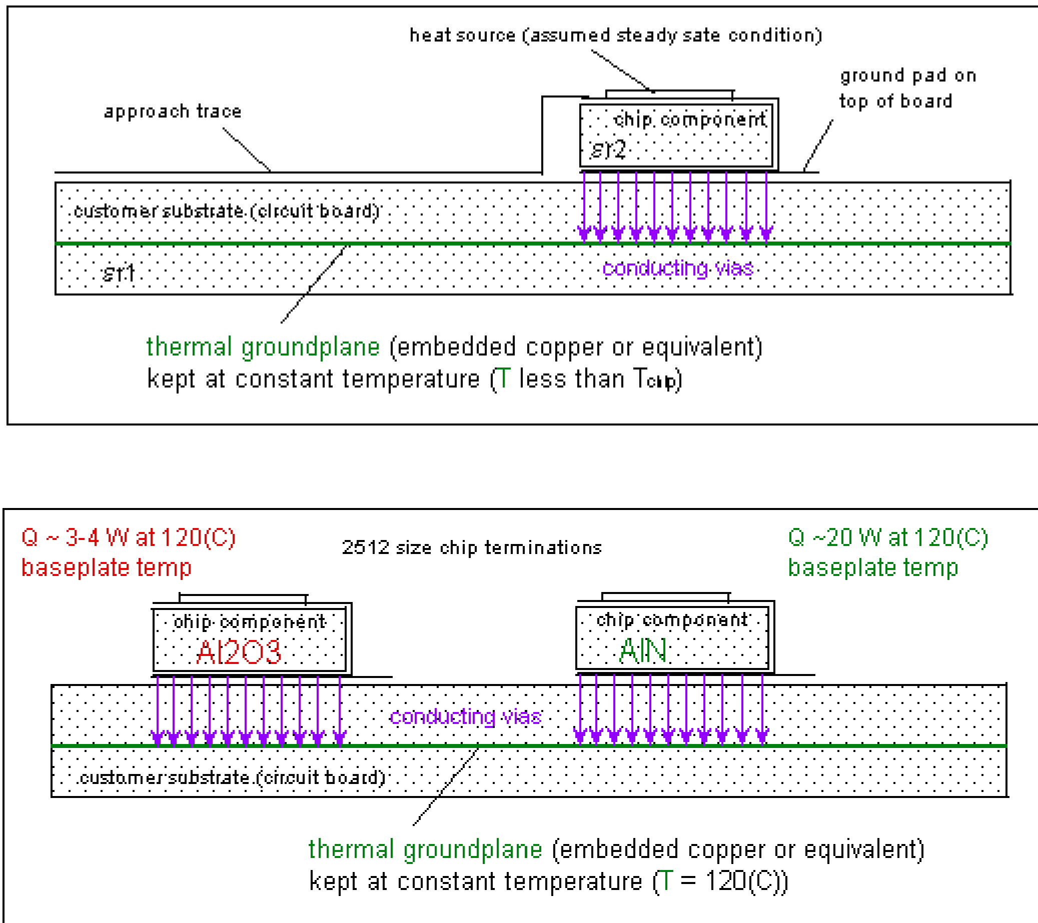


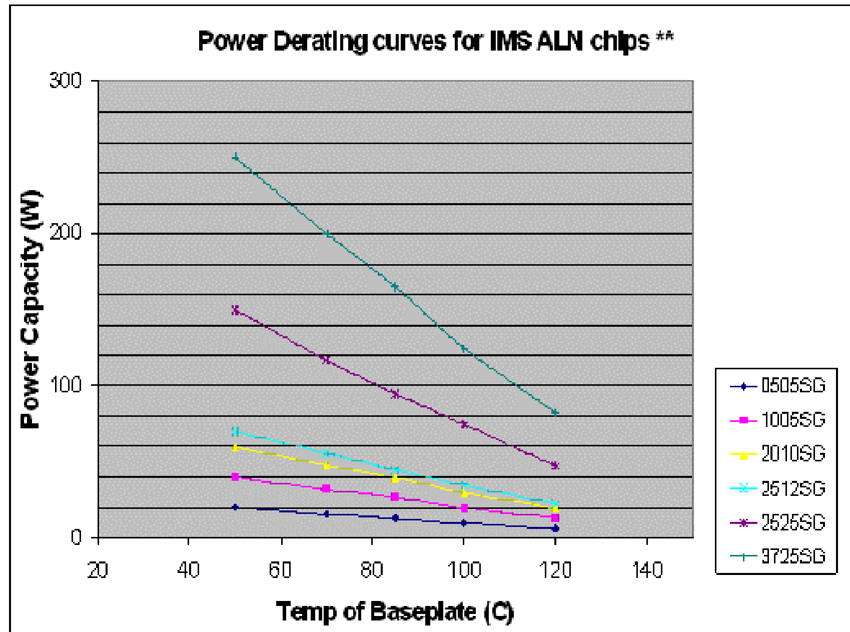
Figure 7: *The benefit of using aluminum nitride (AlN) manifests itself when appropriate thermal management is employed at the circuit level. Without the suggested thermal management, the power capacity of either device would be no more than 1 to 2W.*

Thermal highways on the circuit board

Figure 7 shows a basic method of managing heat at the board level. Chances are, most designers already use vias to connect devices to a mezzanine level for electrical ground. If this plane is kept at a constant temperature, it serves as an excellent heat sink.

Figure 8 shows test results of IMS ALN 50-ohm chip terminations of varying sizes. These devices were tested in a fixture whose design facilitated a constant temperature thermal sink (a solid copper block) that was directly attached to the baseplate of each device under test. Steady-state conditions were assured by allowing the temperatures of the film and baseplate to stabilize. Temperatures were monitored using thermocouples.

The power capacity shown is indicative of the usefulness of high k value ceramics such as ALN.



** 50 ohm resistor terminations

Figure 8

Conclusion

Based on the foregoing arguments and data, it is clear that it is in the circuit designer's best interest to facilitate heat conduction to the highest degree practicable so as to enhance the thermal performance (and thus the overall RF performance) of the SMT devices employed in his/her circuit board, regardless of device substrate material.

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Appendix

Derivation of the temperature function from Equation 2:

As shown previously, the general differential equation for conductive energy transport is:

$$C_p (\delta T/\delta t) = - \text{div} (k \text{ grad } T)$$

or

$$C_p \delta T/\delta t = -\nabla \bullet (k \nabla T) \quad (2)$$

The term on the left is a way of expressing a volumetric heat source Q_v (whose units are W/m^3). The constant C_p is known as the heat capacity per unit volume (assuming constant pressure) whose units are joule/kelvin.

In full "3d" form, this equation looks like this:

$$C_p(\delta T/\delta t) = - [(\delta/\delta x)\underline{i} + (\delta/\delta y)\underline{j} + (\delta/\delta z)\underline{k}] \bullet [k_x(\delta/\delta x)\underline{i} + k_y(\delta/\delta y)\underline{j} + k_z(\delta/\delta z)\underline{k}] T$$

where the first term on the right is the general form for the del operator (∇); the elements $\underline{i}$, $\underline{j}$ and $\underline{k}$ are unit vectors pointing in the directions of x, y and z, respectively and the terms k_x, k_y and k_z represent the generalized constant k for a non-isotropic medium. The multiplication is again indicative of the scalar product so that all cross terms vanish in the above equation. Because the divergence of the gradient of any vector function is equivalent to the second space derivative of that function, the equation simplifies to:

$$C_p \delta T/\delta t = k[\delta^2/\delta x^2+\delta^2/\delta y^2+\delta^2/\delta z^2] T$$

or simply

$$\delta T/\delta t = K \nabla^2 T \quad (3)$$

where T again is the temperature function, K is a constant, which incorporates the constant C_p, and we have once again assumed an isotropic medium. Equation 3 is a special form of the wave equation, similar to that used in quantum mechanical problems. The wave equation also applies to problems involving electrodynamics, optics, sonar, chemical diffusion, etc. We will see that by treating the heat flow problem as a steady state problem, the above equation leads directly to the Laplace equation, which itself will provide a simple solution to Fourier's equation (equation 1) above.

We will once again restrict ourselves to one dimension, rather than three. We expect the solution to equation 2 to have the form of a superposition of two functions, one independent of time and the other dependent on time. This is a reasonable assumption in our case, as the temperature in the ceramic after a long period of time should depend only

on the boundary temperatures (T1 and T2) and the position (x) within the ceramic. The solution then, takes the form of the sum of a steady state solution and a transient solution:

$$T = S(x) + U(x,t) \qquad (4)$$

Where the transient function U(x,t) dies out with increasing time and the function S(x) is dependent on only x. Substitution of equation 4 into 3 yields the following:

$$\delta/\delta t\ U(x,t) = -K\ \nabla^2\ S(x) \qquad (5)$$

($\delta/\delta t$ (S) = 0, since S is not dependent on t)

The form of the transient function (U(x,t)) is generally a product of U(x) with an exponentially decaying function, i.e.:

$$U(x,t) = C_a\ U(x) \exp[-a^2 t]$$

Where C_a and a are other constants.

Over a long time period (i.e., $t = \infty$), the value of the function U(x,t) approaches 0.

Equation 5 thus simplifies to Laplace's equation when transient conditions are eliminated and only steady-state conditions are considered. This means that U(x,t) and its derivative ($\delta/\delta t$ U(x,t)) vanish and equation 5 further simplifies to:

$$\nabla^2\ S(x) = 0 \qquad (6)$$

This is a simple differential equation whose solution is a linear function of x. This helps us realize we are on the right track, for the Fourier equation is also a linear function in x. Equation 6 can be solved by posing a linear form, and referring to the boundary conditions already specified:

$$S_x = C1x + C2 \qquad (7)$$

where C1 and C2 are constants to be determined shortly, but by inspection, must have units of temperature-per-length and temperature, respectively.

Refer again to *Figure 1*. The boundary conditions tell us that S_x must be equal to T1 when x = 0, and S_x must be equal to T2 when x = d. This means that:

T1 = C(0) +C2, from which:

$$C2 = T1 \qquad (8)$$

And, similarly

$$C1(d) + C2 = T2 \text{ (or, since } C2 = T1\text{):}$$

$$C1(d) + T1 = T2$$

Solving for C1, we find:

$$C1 = (T2 - T1)/d \quad (9)$$

By substituting equations 8 and 9 back into 7:

$$S_x = (T2 - T1)\, x/d + T1$$

Taking the gradient of S_x, we have:

$$\delta/\delta x\, S(x) = \nabla S_x = (T2 - T1)/d \quad (10)$$

Now S(x) can be physically equated to our familiar (T_x) in Fourier's relation above, i.e.:

$$S_x \{= (T2 - T1)\, x/d + T1\} = T_x$$

$$\text{So that } \nabla T_x = (T2 - T1)/d$$

This can be substituted back into equation 1 in the main body of the paper to finally obtain:

$$Q/A = -k\,(T2 - T1)/d \quad (11)$$

remembering again that this is for steady state conditions.

Equation 11 can also be obtained by performing a simple integration over the x dimension between x=0 and x=d and from Tx = T1 to Tx = T2, using the same boundary conditions and assuming that variables can be isolated and treated linearly.